Collisional Transport I — Near Equilibrium

How Calculate Transport

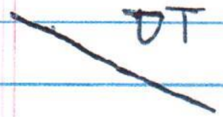
$$\frac{\langle E^2\rangle_{k,\omega}}{8\pi} = \frac{T}{\omega}\,\frac{\mathrm{Im}\,\epsilon}{|\epsilon|^2}$$

shape $\langle f\rangle \Rightarrow$ flctns.

→ Can observe:

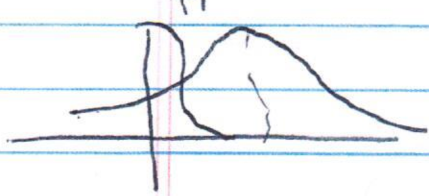
- stable plasma will tend to relax, by return to Maxwellian (global).

i.e.



$\Rightarrow Q = -\chi \nabla T$

↓ collisional diffusion



$\Rightarrow \underline{E} = \eta \underline{J}$

$= \eta(-n_0 e\, \underline{u}_0)$

- process: collisions ⇒ i.e. Coulomb scattering for plasma

- stages: $\begin{cases} 1\ \tau_c \Rightarrow \text{local Maxwellian} \\ \text{Many } \tau_c \Rightarrow \text{relaxation} \end{cases}$

- relaxation ⟺ Entropy Production

i.e. $\dfrac{dS}{dt} \simeq -Q\,\nabla T$

Q ↓ flux; ∇T ↳ force.

→ Theory:

(I)

- Boltzmann Eqn. and H-Thm.

$\Downarrow$

- Landau Eqn. (Boltzmann Eqn. for glancing Coulomb collision)

$\Downarrow$

- Balescu-Lenard Eqn. (Screened Landau Eqn.)

or

(II) - Fokker-Planck Eqn.

$\Downarrow$ Coulomb Collisions (glancing)

- Landau Eqn.

$\Downarrow$ convenient/clever

- Rosenbluth potentials.

(I) For plasma,

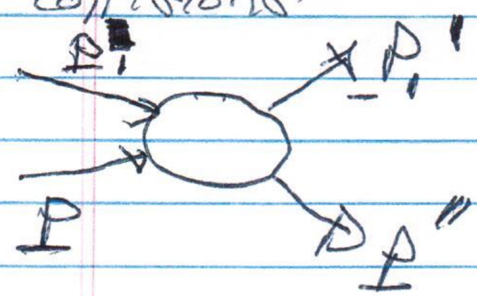
$$\frac{\partial f}{\partial t} + \underline{v}\cdot\underline{\nabla} f + \frac{q}{m}\left(\underline{E} + \cancel{\frac{\underline{v}\times\underline{B}}{c}}\right)\cdot\frac{\partial f}{\partial \underline{v}} = C(f)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \underline{v}\cdot\underline{\nabla} f + \frac{q}{m}\underline{E}\cdot\frac{\partial f}{\partial \underline{v}} = C(f)$$

~ phase space density no longer conserved

~ Origin of C:

- collisions



$\underline{p}$ ≡ 'test' particle

$\underline{p}_1$ ≡ 'background' or "field" particle

⇓

binary collisions

(N.B. what is a 'collision' in sense of TPM)

$W(\underline{p}, \underline{p}_1; \underline{p}_1', \underline{p}') \rightarrow$ transition probability

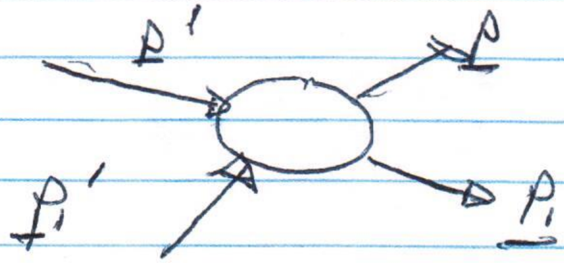
or

- correlations ⇒ residue of BBGKY hierarchy

Convenient to view Boltzmann Eqn. as scattering into/out of state $\underline{p}$:

$$\frac{df}{dt} = C(f) = \text{rate } \underset{\sim}{\text{in}} - \text{rate } \underset{\sim}{\text{out}}$$

in:

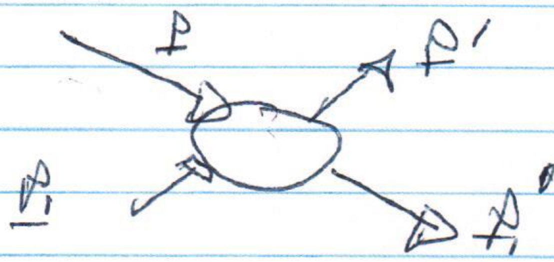


∞

$$\text{in} = \int d\underline{p}' \int d\underline{p}_1' \int d\underline{p}_1 \; W(\underline{p}', \underline{p}_1'; \underline{p}, \underline{p}_1) *$$
$$f(\underline{p}')\, f(\underline{p}_1')$$

and:

out:



$$\text{out} = \int d\underline{p}_1 \int d\underline{p}' \int d\underline{p}_1' \; f(\underline{p})\, f(\underline{p}_1)\, W(\underline{p}, \underline{p}_1; \underline{p}', \underline{p}_1')$$

∞

$$\boxed{\frac{d f(\underline{p})}{dt} = \int d\underline{p}_1 \int d\underline{p}' \int d\underline{p}_1' \; W(\underline{p}, \underline{p}_1; \underline{p}', \underline{p}_1') * \left(f(\underline{p}')\, f(\underline{p}_1') - f(\underline{p})\, f(\underline{p}_1) \right)}$$

→ Boltzmann Equation.

Content:

① → transition probability: contains cross-section, other micro-physics

② ⇒ $f(\underline{p})\, f(\underline{p}_1)$ ⇒ used "Principle of Molecular Chaos" (Boltzmann) to simplify

$$f(\underline{p}, \underline{p}_1) = f(\underline{p})\, f(\underline{p}_1)$$ → $\left\{ \begin{array}{l} \text{probabilities} \\ \text{independent} \end{array} \right.$

⇒ diluteness, again

③ ⇒

$$W(\underline{p}, \underline{p}_1; \underline{p}', \underline{p}_1') = W(\underline{p}', \underline{p}_1'; \underline{p}, \underline{p}_1)$$

"Principle of detailed Balance" ⇒ Probability of Forward Transition = Probability of Back Transition

Further observe:

- one integration trivial, so

$$\underline{p} + \underline{p}_1 = \underline{p}' + \underline{p}_1'$$

- $C(f) = 0$ for $f = f_0$

  i.e. $f_0 = c \exp\left[-\frac{(\epsilon + p \cdot V)}{T} \right]$

  $C \rightarrow 0$, due conservation energy and momentum.

- will show f_0 renders $dS/dt = 0$.

- For $C(f)$ number conserving, must have:

  $C(f) = -\underline{\partial}_p \cdot \underline{J}(p) = -\partial \underline{J} / \partial \underline{p}$

  i.e. $C(f)$ as divergence of flux in momentum space.

This brings us to:

$\boxed{H\text{-Theorem}}$

- a gas / plasma, left alone will evolve to an equilibrium of maximal entropy

- evolution accompanied by entropy production.

- evolution is to uniform Maxwellian
- $dS/dt \geq 0$

for ideal gas

$$S = \int d\underline{x} \int d\underline{p} \; f \ln(e/f)$$

$$\cong \int dx \int dp \left[- f \ln f \right]$$

see notes on entropy, next lecture

Will show $dS/dt \geq 0$.

$\left\{ \begin{array}{l} \text{n.b.} \\ \text{Vlasov Eqn.} \\ \text{cons. entropy} \end{array} \right.$

$$\frac{dS}{dt} = - \int d\Gamma \left[\frac{df}{dt} \ln f + \cancel{f} \frac{1}{\cancel{f}} \frac{df}{dt} \right]$$

$$= - \int d\Gamma \left[C(f) \ln(f) + C(f) \right]$$

$$= - \int d\underline{x} \int d\underline{p} \; \ln(f) \, C(f).$$

→ entropy production due explicitly to collisions

$$= - \int dx \int d\underline{p} \int d\underline{p}_1 \int d\underline{p}' \int d\underline{p}_1' \; \ln f \; w \Big(f(\underline{p}') f(\underline{p}_1') - f(\underline{p}) f(\underline{p}_1) \Big)$$

[redacted]

Lemma

$$\int \varphi(\underline{p})\, C(\underline{p})\, d\underline{p}$$
$$= 1/2 \int d^4\underline{p} \left(\varphi + \varphi_1 - \varphi' - \varphi_1' \right) w f' f_1'$$

where notational shorthand $\ni$

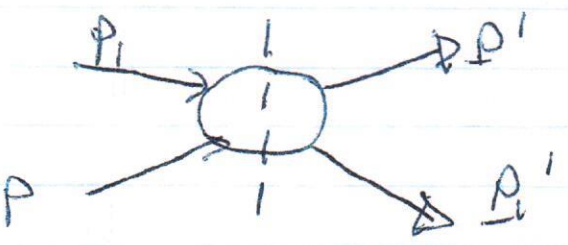
$$d^4\underline{p} = d\underline{p}\, d\underline{p}_1\, d\underline{p}_1'\, d\underline{p}'$$

Now, explicitly:

$$\int d\underline{p}\, \varphi(\underline{p})\, C(f) = \overset{(1)}{\int \varphi\, w(\underline{p}, \underline{p}_1; \underline{p}', \underline{p}_1')\, f' f_1'\, d^4 p}$$
$$\overset{(2)}{- \int \varphi\, w(\underline{p}', \underline{p}_1'; \underline{p}, \underline{p}_1)\, f f_1}$$

now, on (2):

$\rightarrow$ interchange $\underline{p}, \underline{p}_1 \leftrightarrow \underline{p}', \underline{p}_1'$

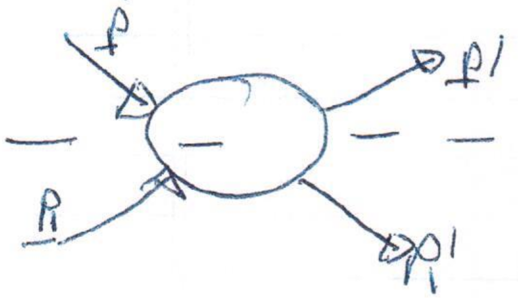


~~rotate~~ flip about !
use T symmetry.

so

$$\int dp\, \varphi\, C(f) = \int d^4p \left\{ \left(\varphi(p) - \varphi(p')\right) w(p, p_1; p', p_1') \times f' f_1' \right\}$$

Now, consider:



and interchange about — — —

i.e. p, p' with p_1, p_1'

n.b. up-down symmetry equivalent

so

$$\int dp\, C(f)\, \varphi =$$

$$= \frac{1}{2} \int d^4p \left\{ \left(\varphi(p) - \varphi(p') + \varphi(p_1) - \varphi(p_1') \right) \times w f' f_1' \right\}$$

this proves Lemma!

Now, let $\varphi = \ln f$,

so Lemma $\Rightarrow$

$$\frac{dS}{dt} = -\frac{1}{2}\int d\underline{x}\int d^4\underline{p}\,\Big(\ln f + \ln f_1 - \ln f' - \ln f_1'\Big) * w\, f' f_1'$$

$$= \frac{1}{2}\int d\underline{x}\int d^4\underline{p}\; w\, f' f_1' \ln\left(f' f_1' / f f_1\right)$$

$x \equiv f' f_1' / f f_1$

$$\boxed{\frac{dS}{dt} = \frac{1}{2}\int d\underline{x}\int d^4\underline{p}\; w\, f f_1\, x \ln x}$$

Now since $\int C(f)\, d\Gamma = 0$

have $\int w f f_1\,(x-1)\, d^4\underline{p}\, d\underline{x} = 0$

i.e. write zero in complex way.

so adding:

$$\boxed{\frac{dS}{dt} = \frac{1}{2}\int d^4p \int dx \; w f f_1 \left[x \ln x - x + 1 \right]}$$

gives entropy production rate.

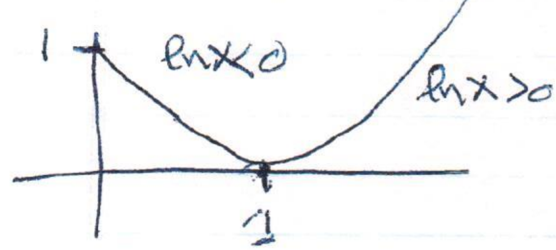
$$F(x) = x \ln x - x + 1$$

$$F' = 1 + \ln x - 1$$

$F(0) = 1$

$F(1) = 0$

so



so

$$\boxed{dS/dt \geq 0}$$

Boltzman H-thm!

- $dS/dt = 0$ for $x = 1$

$$f f_1 = f' f_1'$$

$$\ln f + \ln f_1 = \ln f' + \ln f_1'$$

$\Rightarrow \ln f + \ln f_1 = \text{const.}$

sum of logs conserved in collision

$\Rightarrow \ln f = c + \underline{p} \cdot \underline{V} + \alpha \epsilon$ $\left\{ \text{see next lecture} \right.$

$\alpha < 0$

$\frac{dS}{dt} = 0$ determines Maxwellian

*keys: $\rightarrow$ detailed balance $\leftrightarrow$ w symmetry

$\rightarrow$ molec. chaos
$f(1,2) = f(1) f(2)$

$\rightarrow dS/dt \geq 0$

$dS/dt = 0$ corresponds $C(f) = 0$

collisions drive system to equilibrium

$\rightarrow d\underline{x}$ irrelevant !!!

entropy produced <u>locally</u>

<u>i.e.</u> relaxation to <u>local</u> <u>Maxwellian</u>.

→ Essence of H-thm. is:

* Macroscopic irreversibility from microscopically reversible dynamics +
* molec. chaos (micro-chaos)